

Ore type pseudoquotients

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Abstract

A space of pseudoquotients $\mathcal{B}(X, S)$ is defined as equivalence classes of pairs (x, f) , where x is an element of a non-empty set X , f is an element of S , a commutative semigroup of injective maps from X to X , and $(x, f) \sim (y, g)$ if $gx = fy$. In this note we generalize this construction by replacing the assumption of commutativity of S by Ore type conditions. As in the commutative case, X can be identified with a subset of $\mathcal{B}(X, S)$ and S can be extended to a group of bijections on $\mathcal{B}(X, S)$.

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1 Introduction

The construction of pseudoquotients was introduced in [8] under the name of “generalized quotients.” A space of pseudoquotients, denoted by $\mathcal{B}(X, S)$, is defined as the space of equivalence classes of pairs $(x, f) \in X \times S$, where X is a non-empty set and S is a commutative semigroup of injective maps from X to X . The equivalence relation is defined as follows: $(x, f) \sim (y, g)$ if $gx = fy$. It is a generalization of the constructions of the field of quotients from an integral domain. For example, if we take $X = \mathbb{Z}$ and $S = \mathbb{N}$ acting on \mathbb{Z} by multiplication, then $\mathcal{B}(\mathbb{Z}, \mathbb{N}) = \mathbb{Q}$.

Pseudoquotients have desirable properties. The set X can be identified with a subset of $\mathcal{B}(X, S)$ and the semigroup S can be extended to a commutative group of bijections acting on $\mathcal{B}(X, S)$ (see, for example, [6]). Under natural conditions on S , the algebraic structure of X extends to $\mathcal{B}(X, S)$. There are indications that pseudoquotients can be useful in analysis (see, for example, [1],

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[2], or [3]). However, the assumption of commutativity of S can be a serious limitation. In [11] we find an interesting application of the standard noncommutative localization in the theory of integro-differential algebras and operators. Noncommutative methods in generalized functions are also considered in [5] and [4]. The construction presented in this note will make it possible to use similar methods in situations where the standard theory does not apply.

In this note we present a generalization of the construction of pseudoquotients where commutativity of S is replaced by the left Ore condition. We obtain a space $\mathcal{B}(X, S)$ that contains a copy of X . If, in addition to the left Ore condition, we assume right cancellation in S , then S can be extended to a group of bijections acting on $\mathcal{B}(X, S)$. While our construction is similar to the left Ore localization of a ring (see, for example, [7]), we do not require any structure on the set X . The approach used in this note is similar to the one used in [10] where X is assumed to be an A -monoid and S is a localizable submonoid of $\text{End}_{\mathcal{M}}(M)$.

The main motivation for considering this generalization is applications to generalized functions and related ideas [9]. However, since the construction is very general, it can be used as a tool in other areas of mathematics.

2 Pseudoquotients

Let X be a nonempty set and let S be a semigroup acting on X injectively. We will assume that S satisfies the following Ore condition:

⓪ For any $f, g \in S$ there exist $f', g' \in S$ such that $f'g = g'f$.

In $X \times S$ we introduce an equivalence relation:

$(x, f) \sim (y, g)$ if there exist $f', g' \in S$ such that $f'g = g'f$ and $f'y = g'x$.

It is easy to verify that \sim is reflexive and symmetric (even without ⓪). To show that it is also transitive we assume $(x, f) \sim (y, g)$ and $(y, g) \sim (z, h)$. Let $f', g', g'', h'' \in S$ be such that

$$f'g = g'f, f'y = g'x, g''h = h''g, \text{ and } g''z = h''y.$$

By ⓪, there exist $j, k \in S$ such that $jf' = kh''$. Then

$$jg'f = jf'g = kh''g = kg''h$$

and

$$jg'x = jf'y = kh''y = kg''z.$$

Consequently, $(x, f) \sim (z, h)$.

We define $\mathcal{B} = \mathcal{B}(X, S) = (X \times S) / \sim$. Elements of $\mathcal{B}(X, S)$ will be called pseudoquotients. We will use the standard notation to denote elements of \mathcal{B} : $[(x, f)] = \frac{x}{f}$.

Lemma 2.1. *For all $x \in X$ and $f, g \in S$ we have*

- (a) $(fx, f) \sim (gx, g)$,
- (b) *If $(fx, f) \sim (y, g)$, then $y = gx$,*
- (c) *If $(fx, f) \sim (fy, f)$, then $x = y$.*

Proof. (a) By \mathbb{O} , there are $f', g' \in S$ such that $f'g = g'f$. Then also $f'gx = g'fx$.

(b) If $(fx, f) \sim (y, g)$, then there are $f', g' \in S$ such that $g'fx = f'y$ and $g'f = f'g$. Hence

$$f'y = g'fx = f'gx.$$

Since f' is injective, we obtain $y = gx$.

(c) follows from (b), if we take fy in place of y and f in place of g and then use injectivity of f . \square

Part (a) of Lemma 2.1 implies that the map $\iota : X \rightarrow \mathcal{B}$ defined by

$$\iota(x) = \frac{fx}{f}$$

is well-defined. From part (b) we have $[(fx, f)] = \{(gx, g) : g \in S\}$. This, combined with (c), implies that ι is injective.

Pseudoquotients have many properties of standard quotients. The following simple lemma gives an example of such a property.

Lemma 2.2. *For all $x, y \in X$ and $f, g, h \in S$ we have*

- (a) *If $(x, f) \sim (y, g)$, then $(x, f) \sim (hy, hg)$,*
- (b) $\frac{x}{f} = \frac{gx}{gf}$.

Proof. (a) If $(x, f) \sim (y, g)$, then there exist $f', g' \in S$ such that $g'x = f'y$ and $g'f = f'g$. If $h \in S$, then $\alpha f' = \beta h$ for some $\alpha, \beta \in S$, by \mathbb{O} . Hence

$$\alpha g'x = \alpha f'y = \beta hy$$

and

$$\alpha g'f = \alpha f'g = \beta hg.$$

(b) is a direct consequence of (a). \square

3 Extendability of maps

One of the fundamental properties of pseudoquotients with commutative S is that S can be extended to a group of bijections acting on \mathcal{B} . To obtain such a result without commutativity, in addition to \mathbb{O} , we will assume right cancellation for S , that is:

Ⓐ If $f_1g = f_2g$, where $f_1, f_2, g \in S$, then $f_1 = f_2$.

Lemma 3.1. *If $f'g = g'f$ and $f''g = g''f$, then $hf'' = kf'$ and $hg'' = kg'$ for some $h, k \in S$.*

Proof. Let $f'g = g'f$ and $f''g = g''f$. Then there are $h, k \in S$ such that $hf'' = kf'$, by ①. Hence, $hf''g = kf'g$ and $hg''f = kg'f$, which implies $hg'' = kg'$ by Ⓐ. \square

Lemma 3.2. *Let $f'g = g'f$ and $f''g = g''f$. If $f'y = g'x$ for some $x, y \in S$, then $f''y = g''x$.*

Proof. Let $h, k \in S$ be as defined in Lemma 3.1. Then we have

$$hf''y = kf'y = kg'x = hg''x.$$

Since h is injective, we obtain $f''y = g''x$. \square

Note that from the above lemma it follows that, if $\frac{x}{f} \sim \frac{y}{g}$, then we have $g'x = f'y$ for any $g', f' \in S$ such that $f'g = g'f$.

Lemma 3.3. *If $f'g = g'f$ and $f''g = g''f$, then $\frac{g'x}{f'} = \frac{g''x}{f''}$ for all $x \in X$.*

Proof. Let h, k be defined as in Lemma 3.1. Then $\frac{g'x}{f'} = \frac{g''x}{f''}$ because $kf' = hf''$ and $kg'x = hg''x$. \square

Theorem 3.4. *A function $g \in S$ can be extended to a function $\tilde{g} : \mathcal{B} \rightarrow \mathcal{B}$ by*

$$\tilde{g}\frac{x}{f} = \frac{g'x}{f'},$$

where $f', g' \in S$ are any functions such that $f'g = g'f$.

Proof. By Lemma 3.3, $\frac{g'x}{f'}$ is independent of choice of g', f' , as long as $f'g = g'f$.

Now we show that, if $\frac{x_1}{f_1} = \frac{x_2}{f_2}$, then $\tilde{g}\frac{x_1}{f_1} = \tilde{g}\frac{x_2}{f_2}$. Let $f'_1, f'_2, g', g'' \in S$ be such that

$$g'f_1 = f'_1g \text{ and } g''f_2 = f'_2g.$$

We need to show that $\frac{g'x_1}{f'_1} \sim \frac{g''x_2}{f'_2}$. Let $f''_1, f''_2 \in S$ be such that $f'_1f''_2 = f'_2f''_1$. Then

$$f''_1g''f_2 = f''_1f'_2g = f''_2f'_1g = f''_2g'f_1.$$

Since $\frac{x_1}{f_1} = \frac{x_2}{f_2}$, we obtain $f''_1g''x_2 = f''_2g'x_1$, by Lemma 3.2.

We have shown that \tilde{g} is well defined. Finally we show that \tilde{g} is an extension of g . If $f'g = g'f$, then $f'gx = g'fx$ and hence

$$\tilde{g}\iota(x) = \tilde{g}\frac{fx}{f} = \frac{g'fx}{f'} = \frac{f'gx}{f'} = \iota(gx).$$

\square

Lemma 3.5. $\tilde{g} : \mathcal{B} \rightarrow \mathcal{B}$ is bijective for every $g \in S$.

Proof. Injectivity: Suppose $\tilde{g}(\frac{x_1}{f_1}) = \tilde{g}(\frac{x_2}{f_2})$ and let $f'_1, g', f'_2, g'' \in S$ be such that

$$f'_1 g = g' f_1 \quad \text{and} \quad f'_2 g = g'' f_2.$$

Then $\tilde{g}(\frac{x_1}{f_1}) = \frac{g' x_1}{f'_1}$ and $\tilde{g}(\frac{x_2}{f_2}) = \frac{g'' x_2}{f'_2}$. Now let $h_1, h_2 \in S$ be such that $h_2 f'_1 = h_1 f'_2$. Then $h_2 g' x_1 = h_1 g'' x_2$, by Lemma 3.2. We have

$$\frac{x_1}{f_1} = \frac{g' x_1}{g' f_1} = \frac{g' x_1}{f'_1 g} = \frac{h_2 g' x_1}{h_2 f'_1 g}$$

and

$$\frac{x_2}{f_2} = \frac{g'' x_2}{g'' f_2} = \frac{g'' x_2}{f'_2 g} = \frac{h_1 g'' x_2}{h_1 f'_2 g}.$$

Since $\frac{h_1 g'' x_2}{h_1 f'_2 g} = \frac{h_2 g' x_1}{h_2 f'_1 g}$, we obtain $\frac{x_1}{f_1} = \frac{x_2}{f_2}$ by transitivity.

Surjectivity: If $\frac{x}{f} \in \mathcal{B}$, then $\frac{x}{f} = \frac{fx}{f^2} = \tilde{g}(\frac{x}{fg})$ since $f(fg) = f^2(g)$. \square

Note that $\tilde{g}^{-1}(\frac{x}{f}) = \frac{x}{fg}$.

Theorem 3.6. S can be extended to a group G of bijections acting on \mathcal{B} . Moreover, G satisfies conditions \mathbb{O} and \mathbb{A} .

Proof. It suffices to show that G satisfies conditions \mathbb{O} and \mathbb{A} , but this follows immediately from the fact that everything in G is invertible. \square

Remark 3.7. We can think of \mathcal{B} as the set of all solutions ξ of the equations $f(\xi) = x$, for any $f \in S$ and $x \in X$. These solutions are unique, for if $\tilde{f}(\frac{x_1}{f_1}) = x$ and $\tilde{f}(\frac{x_2}{f_2}) = x$, then $\frac{x_1}{f_1} = \frac{x_2}{f_2}$, since \tilde{f} acts injectively on \mathcal{B} . Also any element of \mathcal{B} is a solution to some equation, since $\tilde{f}(\frac{x}{f}) = \frac{fx}{f^2} = \iota(x)$.

4 Simple examples

We end this paper with some simple examples of the described construction.

Example 4.1. Let $X = \mathbb{N}$ and let S be the semigroup of all functions on \mathbb{N} of the form $f(x) = mx^n$ where $m, n \in \mathbb{N}$. It is easy to see that \mathbb{O} and \mathbb{A} are satisfied. In this case \mathcal{B} can be identified with the set of all real numbers of the form $\sqrt[n]{\frac{k}{m}}$, where $k, m, n \in \mathbb{N}$, and the extended group G can be described as all functions of the form $f(x) = px^q$, where p and q are positive rational numbers.

Example 4.2. Let $X = \mathbb{Z}^n$ and let S be the semigroup of all functions $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ of the form $f(x) = Mx + b$, where M is an $n \times n$ matrix of rank n with integer entries and $b \in \mathbb{Z}^n$. To check \mathbb{O} , let $f(x) = M_1 x + b_1$ and $g(x) = M_2 x + b_2$. If $m_1 = \det M_1$ and $m_2 = \det M_2$, then the functions

$$f'(x) = m_1 m_2 M_2^{-1} x + m_1 m_2 M_1^{-1} b_1$$

and

$$g'(x) = m_1 m_2 M_1^{-1} x + m_1 m_2 M_2^{-1} b_2$$

satisfy the equation $f'g = g'f$. It is easy to see that \mathbb{A} is also satisfied.

Here \mathcal{B} can be identified with \mathbb{Q}^n and the extended group G can be described as all functions $f : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$ of the form $f(x) = Mx + b$, where M is an $n \times n$ invertible matrix with rational entries and $b \in \mathbb{Q}^n$.

Example 4.3. Let

$$X = \left\{ \sum_{k=0}^n \lambda_k \chi_{[k, k+1)} : \lambda_k \in \mathbb{R} \text{ and } n \in \mathbb{N} \right\}$$

and let $\delta, \tau : X \rightarrow X$ be defined as follows:

$$\delta \left(\sum_{k=0}^n \lambda_k \chi_{[k, k+1)} \right) = \sum_{k=0}^n \frac{\lambda_k}{2} \chi_{[2k, 2k+2)} \quad \text{and} \quad \tau \left(\sum_{k=0}^n \lambda_k \chi_{[k, k+1)} \right) = \sum_{k=0}^n \lambda_k \chi_{[k+1, k+2)}.$$

If we define $I = \chi_{[0, 1)}$, then we can write

$$\sum_{k=0}^n \lambda_k \chi_{[k, k+1)} = \sum_{k=0}^n \lambda_k \tau^k I.$$

Let S be the semigroup generated by δ and τ . Since $\delta\tau = \tau^2\delta$, we have $S = \{\tau^m \delta^n : m, n \geq 0\}$. Since δ and τ are injective, S acts on X injectively. We will show that S satisfies \mathbb{O} and \mathbb{A} . To verify \mathbb{O} , assume $\tau^{m_1} \delta^{n_1}, \tau^{m_2} \delta^{n_2} \in S$. If $n_1 = n_2 = n$, then

$$\tau^{m_2} \tau^{m_1} \delta^n = \tau^{m_1} \tau^{m_2} \delta^n.$$

If $n_1 < n_2$, then

$$\tau^{m_2} \delta^{n_2 - n_1} \tau^{m_1} \delta^{n_1} = \tau^{2m_1} \tau^{m_2} \delta^{n_2}.$$

To verify \mathbb{A} , first observe that for every $\phi \in S$ the representation $\phi = \tau^m \delta^n$ is unique. Now, if

$$\tau^{m_1} \delta^{n_1} \tau^k \delta^l = \tau^{m_2} \delta^{n_2} \tau^k \delta^l,$$

we have

$$\tau^{m_1 + 2k} \delta^{n_1 + l} = \tau^{m_2 + 2k} \delta^{n_2 + l}.$$

Thus $m_1 = m_2$ and $n_1 = n_2$.

Note that, since δ and τ preserve the integral and the L^1 -norm, we can define

$$\int \frac{f}{\tau^m \delta^n} = \int f \quad \text{and} \quad \left\| \frac{f}{\tau^m \delta^n} \right\|_1 = \|f\|_1,$$

where $f \in X$. The obtained space of pseudoquotients \mathcal{B} can be identified with a dense subspace of $L^1(\mathbb{R})$.

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